

Correlation functions of Polyakov loops at tree level

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Abstract

We compute the correlation functions of Polyakov loops in $SU(N_c)$ gauge theories by explicitly summing all diagrams at tree level in two special cases, for $N_c = 2$ and $N_c = \infty$. When $N_c = 2$ we find the expected we find Coulomb-like behavior at short distances, $\sim 1/x$ as the distance $x \rightarrow 0$. In the planar limit at $N_c = \infty$ we find a weaker singularity, $\sim 1/\sqrt{x}$ as $x \rightarrow 0$. In each case, at short distances the behavior of the correlation functions between two Polyakov loops, and the corresponding Wilson loop, are the same. We suggest that such non-Coulombic behavior is an artifact of the planar limit.

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I. INTRODUCTION

In gauge theories at a nonzero temperature T , the order parameter for deconfinement in a $SU(N_c)$ gauge theory is the Polyakov loop,

$$L(x) = \frac{1}{N_c} \text{tr} \mathcal{P} \exp \left(ig \int_0^{1/T} A^0(x, \tau) d\tau \right) . \quad (1)$$

This loop is related to the propagator of an infinitely massive test quark, which sits at a spatial point x , and just propagates forward in imaginary time, $\tau : 0 \rightarrow 1/T$. A quantity invariant under small gauge transformations is obtained by tracing over color, denoted by tr ; \mathcal{P} represents path ordering in τ , g is the gauge coupling, and A^0 the vector potential in the τ direction.

To be definite, we consider the case of a test particle, and its associated loop, in the fundamental representation of a $SU(N_c)$ gauge theory. (As for test particles, though, there are (independent) loops for all irreducible representations.) For two colors the loop is real, but for three or more colors the anti-loop, $L^*(x)$, is an independent quantity, representing the propagation of an anti-particle backwards in (imaginary) time.

The correlation functions of Polyakov loops are fundamental to our understanding of the behavior of gauge theories in a thermal bath. These have been studied in great detail, both perturbatively, and from numerical simulations on the lattice [1–5]. In general, understanding the behavior of the loop correlation functions is involved, requiring the use of effective Lagrangians and non-perturbative input, such as from numerical simulations on the lattice [6].

This is not true in an Abelian gauge theory, where at least at tree level, one can immediately compute the correlation function between two loops. In this case, the path need not be straight but can be arbitrary. A source J^μ couples to the photon as $\sim \exp(i \oint ds J^\mu A_\mu)$, where “ s ” the worldline of the test particle. It is then trivial to integral over over A_μ ,

$$\sim \exp \left(-\frac{e^2}{2} \oint ds \oint ds' J^\mu \Delta_{\mu\nu} J^\nu \right) , \quad (2)$$

and obtain the sources tied together by a photon propagator, $\Delta_{\mu\nu}$.

This exercise is not possible in a non-Abelian gauge theory, where the fields are matrices and do not commute. As an exercise, in this paper we compute the correlation functions between Polyakov loops at tree level. Even in this instance the generalization of Eq. (2)

is not possible for arbitrary N_c . This is basically because for $3 \leq N_c < \infty$, one cannot determine the element of the Lie group, the Wilson line, from that of the Lie algebra, A^0 . We compute the two point functions of loops in two special limits, for two colors, and in the planar limit, $N_c \rightarrow \infty$ with $g^2 N_c$ held fixed.

As a computation at tree level, our results are valid only in the limit when the temperature T is very large, so that the running coupling constant, $g^2(T)$, is small. We compute the correlation functions between two loops at an arbitrary distance, x ; this will always appear with the only dimensional scale in the problem, the temperature T , as xT . At large distances, this must agree with the results of an ordinary perturbative expansion computed to tree level. Because each loop is a trace over color indices, the leading diagram, from one loop gluon exchange, cancels, and so unlike the Abelian theory, there is no term $\sim g^2/(xT)$. The leading diagram is given by two gluon exchange, which is $\sim g^4/(xT)^2$. This is valid at both $N_c = 2$ and $N_c = \infty$, and is a trivial check on our results.

The natural expansion parameter in the correlation functions is $\sim g^2/(xT)$. Thus at short distances, naive perturbation theory cannot be used. When the loops are very close together, though, $xT \ll 1$, though, physical intuition suggests that it shouldn't matter how the loops are tied off from one another. Thus the result should be the same as for a Wilson loop that is very long. As a Wilson loop is a single trace over color, single gluon exchange contributes, so the result at small distances should be Coulombic, $\sim g^2/(xT)$.

We find that at short distances, in both cases the leading behavior at short distances is the same for the correlation between a Polyakov loop and an anti-loop (or two Polyakov loops), and the corresponding Wilson loop. For two colors, at short distances the behavior is Coulomb-like, although the coefficient is not identical to that from single gluon exchange. For an infinite number of colors, the behavior is milder, $\sim \sqrt{g^2/(xT)}$. Although our results are only valid to tree level, for arbitrary distances we obtain non-trivial functions of $g^2/(xT)$ which can be written in closed form.

In the Conclusions, Sec. (V), we discuss why the non-Coulombic behavior at short distances is an artifact of the planar limit, and discuss how to compute the short distance behavior from effective theories [3, 4]. Our study was inspired by recent studies of the susceptibilities of Polyakov loops by Lo, Friman, Kaczmarek, Redlich, and Sasaki [5]. We also discuss in the Conclusions the implications of our analysis for the measurements of such loop susceptibilities.

II. CONVENTIONS

Since we only compute Polyakov and Wilson loops, at the outset it is convenient to adopt static gauge, $\partial_0 A^0(\tau, \vec{x}) = 0$. It is also necessary to fix the gauge dependence of spatial gluons, A^i , at a given time, but at tree level the spatial gluons do not enter into the correlators of Polyakov loops or of the Wilson loops which we consider.

The advantage of static gauge is that for the Polyakov loop, time ordering can be ignored, with the Polyakov loop just the exponential of a single field $A^0(\vec{x})$. For single gluon exchange, it is clear that in an arbitrary gauge, time ordering gives the same result as in static gauge. It is not so obvious for the exchange of two or more gluons, but in the end, both Polyakov (and Wilson) loops are gauge invariant. In any case, what we are really interested in is how the correlator of the exponential of non-Abelian A^0 fields is related to the correlator of the A^0 fields.

The gluon field $A^0 = A_a^0 t^a$, normalizing the generators as $\text{tr}(t^a t^b) = \delta^{ab}/2$, $a, b = 1 \dots N_c^2 - 1$. For any N_c , in static gauge the gluon propagator for A_a^0 is color diagonal,

$$\Delta_{00}^{ab}(\vec{x}) = \langle A_a^0(\vec{x}) A_b^0(0) \rangle = \delta^{ab} T \Delta(\vec{x}) \quad , \quad \frac{1}{T} \Delta(\vec{x}) = \frac{1}{T} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i \vec{k} \cdot \vec{x}}}{\vec{k}^2} = \frac{1}{4\pi} \frac{1}{x T} . \quad (3)$$

Since we only compute two point functions, which are a function of a single spatial distance \vec{x} , henceforth we replace \vec{x} by $x = |\vec{x}|$. We introduce $\Delta(x)/T$ since Polyakov loops are dimensionless, and so at tree level their correlation functions are functions only of xT .

As our computation is only at tree level, several well known physical effects do not enter. Beyond leading order, the Debye mass, $\sim gT$, arises to screen correlation functions involving A^0 . Also, the renormalization mass scale enters to ensure the coupling constant runs.

The virtue of computing at tree level is that it is mathematically well defined. Of course then one must be careful to consider the limitations of this approximation, as we discuss in the Conclusions, Sec. (V).

III. TWO COLORS

For two colors, we can use the well known property of Pauli matrices, σ^a , and evaluate any element of the Lie group from that of the Lie algebra,

$$\exp(i \beta^a \sigma^a) = \cos(\beta) + i \hat{\beta}^a \sigma^a \sin(\beta) \quad , \quad (4)$$

where $\beta^a = \widehat{\beta}^a \beta$, $(\widehat{\beta})^2 = 1$.

The vector potential is $A^0 = A_a^0 \sigma^a / 2$. Since A_a^0 are the only degrees of freedom which enter, we denote it simply as A_a , and use a convention for repeated indices in color,

$$A_a^2 = \sum_{a=1}^3 (A_a)^2. \quad (5)$$

Hence the Polyakov loop is given by the first term, from the cosine:

$$L(x) = \frac{1}{2} \text{tr} \left[e^{ig A_a^0(x) \sigma^a / (2T)} \right] = \cos \left(\frac{g}{2T} \sqrt{A_a(x)^2} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{g^2}{4T^2} A_a(x)^2 \right)^n \quad (6)$$

This also shows that for two colors the loop is always real, $L = L^*$. The correlation function of two loops is then

$$\langle L(x) L(0) \rangle = \left\langle \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{g^2}{4T^2} A_a(x)^2 \right)^n \sum_{n'=1}^{\infty} \frac{(-1)^{n'}}{(2n')!} \left(\frac{g^2}{4T^2} A_b(0)^2 \right)^{n'} \right\rangle \quad (7)$$

We ignore self energy corrections, which vanish with dimensional regularization. Then the correlator only receives contributions from terms where $n = n'$,

$$\langle L(x) L(0) \rangle = \sum_{n=0}^{\infty} \frac{1}{((2n)!)^2} \left(\frac{g}{2T} \right)^{4n} \langle (A_a(x)^2)^n (A_a(0)^2)^n \rangle. \quad (8)$$

If only one color contributed, then this correlation function would follow immediately. Because for two colors there are three gluons, we have to solve the problem of how many ways to tie three fields together in all possible ways. We do this in two ways, as a cross check on each method. The first method is combinatoric, just a matter of counting different fields together. The second is analytic, and may be useful for other problems, when counting permutations might be confusing.

A. Combinatoric analysis

We solve a more general problem, that of computing the combinatoric factor in Eq. (8) when the sum over fields is from $a = 1$ to m , instead of $m = 3$. We work inductively, starting from $m = 1$, where the answer is immediate, and then generalize to arbitrary m by induction.

The problem is to determine the number of ways in which we can tie the fields

$$\langle (A_a(x)^2)^n (A_b(0)^2)^n \rangle \quad (9)$$

together. We denote this as $\mathcal{D}(2n, m)$.

For a single component field, $m = 1$, the answer is obvious. The first $A(x)$ can connect with any of $2n$ $A(0)$'s; the second $A(x)$ can connect to any of the remaining $2n - 1$ $A(0)$'s, and so on. This gives the usual factorial, $\mathcal{D}(2n, 1) = (2n)(2n - 1) \dots = (2n)!$.

To solve for $m > 1$, we start with some special cases.

When $n = 1$, the first $A_a(x)$ can connect to one of the two $A_b(0)$'s. The second $A_a(x)$ must then connect to the other $A_b(0)$. Since $a = b$, the sum over a gives m , and so

$$\mathcal{D}(2, m) = 2! m . \quad (10)$$

Now consider $n = 2$. The first $A_a(x)$ can connect to any of the four $A_b(0)$'s. There are then two possibilities. The second $A_a(x)$ can connect to the partner of the first $A_b(0)$, which gives a factor of m . Tying the second pair together gives $\mathcal{N}(1, m) = 2m$, or $8m^2$ in all.

Alternately, the second $A_a(x)$ can tie to a different pair, $A_c(0)^2$. It can do so in one of two ways. This leaves two ways for the third $A_d(x)$ to connect to one of the two remaining $A(0)$'s. All four A 's have the same index, so there is one sum over a , or $16m$ in all. The sum is $8m^2 + 16m$, or

$$\mathcal{D}(2, m) = 4!! (m + 2)m . \quad (11)$$

where $n!! = n(n - 2) \dots$.

For any n , the numbers of ways to connect with the highest powers of m is evident, just m^n . The first $A_a(x)$ can connect with any of the $2n$ $A_b(0)$'s. The second $A_a(x)$ has to connect with the pair of $A_b(0)$, which gives $2n \times m$. The third $A_c(x)$ can connect to any of the remaining $(2n - 2)$ $A_d(0)$'s, and so on. Thus for large m , the leading term is

$$\mathcal{D}(2n, m \rightarrow \infty) = (2n) m (2n - 2) m \dots 2 m = (2n)!! m^n . \quad (12)$$

Looking at the results for $n = 1$ and 2 , and the result for large m , a guess for arbitrary m and n is

$$\mathcal{D}(2n, m) = (2n)!! \frac{(2n - 2 + m)!!}{(m - 2)!!} . \quad (13)$$

When $m = 1$ we define $(-1)!! = 1$, so $\mathcal{D}(2n, 1) = (2n)!!(2n - 1)!! = (2n)!$.

To establish the result for $m > 1$ we use induction, and consider how $\mathcal{D}(2n + 2, m)$ is related to $\mathcal{D}(2n, m)$. As always, the first $A_a(x)$ can connect to $2n + 2$ different $A_b(0)$'s. There are two ways in which the second $A_a(x)$ can connect to a field $A_b(0)$. The simplest is

if it connects to the pair of the $A_b(0)$ which the first $A_a(x)$ connected to. There is only one such $A_b(0)$ to connect to. The sum over $a = b$ gives a factor of m , so we have $(2n + 2)m$ times the number of ways of tying n pairs of A_a together, which is $\mathcal{D}(2n, m)$.

Alternately, the second $A_a(x)$ can tie with a $A_c(0)$, which is in a *different* pair than the first $A_a(x)$ connected to. There are $2n$ different $A_c(0)$'s to choose from. This possibility looks complicated at first, but explicitly what we have is

$$\langle A_a(x)A_b(0) \rangle \langle A_a(x)A_c(0) \rangle \left\langle (A_c(x)^2)^n A_b(0)A_c(0) (A_d(0)^2)^{n-1} \right\rangle . \quad (14)$$

Now $A_b(0)$ and $A_c(0)$ start out as being in different pairs. However, the propagators are diagonal in the color indices, and evaluating the propagators in Eq. (14) gives $a = b$ and $b = c$. After doing so, we have that $A_b(0)$ and $A_c(0)$ have the same indices. There is no factor of m , because we still have to tie $A_b(0)$ and $A_c(0)$ to other $A(x)$'s. Consequently, we are left with $(2n + 2)(2n)$ times the number of ways of tying n pairs together, which is $\mathcal{D}(2n, m)$.

The sum of these two terms is

$$\mathcal{D}(2n + 2, m) = (2n + 2) (2n + m) \mathcal{D}(2n, m) . \quad (15)$$

The solution to Eq. (10), for $n = 2$, Eq. (11), for $n = 4$, and this equation, is the general result of Eq. (13).

For the case of interest, $m = 3$, we obtain an especially simple result,

$$\mathcal{D}(2n, 3) = (2n)!! (2n + 1)!! = (2n + 1)! . \quad (16)$$

With this symmetry factor in hand, from Eq. (8) the two point function of loops

$$\langle L(x)L(0) \rangle = \sum_{n=0}^{\infty} \frac{(2n + 1)!}{((2n)!)^2} \left(\frac{g^2}{4T} \Delta \right)^{2n} = \cosh(z) + z \sinh(z) \quad , \quad z = \frac{g^2}{16\pi} \frac{1}{xT} . \quad (17)$$

At large distances, $z \rightarrow 0$ or $x \rightarrow \infty$,

$$\langle L(x)L(0) \rangle \approx 1 + \frac{3}{2} z^2 + \frac{5}{24} z^4 + \dots \approx 1 + \frac{3}{32} \left(\frac{g^2}{4\pi} \right)^2 \frac{1}{(xT)^2} + \frac{5}{24} \left(\frac{g^2}{16\pi xT} \right)^4 + \dots . \quad (18)$$

For general N_c , to $\sim g^4$ [3, 4]

$$\langle L^*(x)L(0) \rangle \approx 1 + \frac{N_c^2 - 1}{8N_c^2} \left(\frac{g^2}{4\pi} \right)^2 \frac{1}{(xT)^2} + \dots \quad , \quad x \rightarrow \infty . \quad (19)$$

This result is due to the exchange of two gluons, with the factor of $(N_c^2 - 1)/(2N_c^2)$ from the Casimir for the fundamental representation. This agrees with Eq. (18) for $N_c = 2$. Notice that the term $\sim z^4 \sim g^8$ is of higher order than has been computed previously. This is because we only include tree diagrams.

At short distances, $z \rightarrow \infty$,

$$\langle L(x)L(0) \rangle \approx \frac{z}{2} \exp(z) \approx \frac{1}{2} \exp \left(+ \frac{g^2}{16\pi} \frac{1}{xT} + \log \left(\frac{g^2}{16\pi xT} \right) \right) , \quad x \rightarrow 0 . \quad (20)$$

The coefficient in the exponential is the same as for the Wilson loop at short distances, Eq. (41) in Sec. (III C).

What is novel here is that Eq. (17) is valid for arbitrary distances, and interpolates smoothly between small and large values of $g^2/(xT)$.

B. Analytic method

We work with components, and use the multinomial theorem

$$(A_a(x)^2)^n = \sum_{k_1, k_2, k_3=1}^n \frac{n!}{k_1! k_2! k_3!} A_1^{2k_1}(x) A_2^{2k_2}(x) A_3^{2k_3}(x) \delta_{k_1+k_2+k_3, n} . \quad (21)$$

After expanding each bracket in Eq. (8) we ignore self energy correction to match the powers of A^0 ,

$$\begin{aligned} \langle L(x)L(0) \rangle &= \sum_{n=0}^{\infty} \frac{1}{((2n)!)^2} \left(\frac{g}{2T} \right)^{4n} \\ &\times \sum_{k_1, k_2, k_3=0}^n \left(\frac{n!}{k_1! k_2! k_3!} \right)^2 \langle A_1^{2k_1}(x) A_2^{2k_2}(x) A_3^{2k_3}(x) A_1^{2k_1}(0) A_2^{2k_2}(0) A_3^{2k_3}(0) \rangle \delta_{k_1+k_2+k_3, n} . \end{aligned} \quad (22)$$

Since we have evaluated the fields component by component, it is direct to evaluate the correlation functions,

$$\begin{aligned} &\langle A_1(x)^{2k_1} A_2(x)^{2k_2} A_3(x)^{2k_3} A_1(0)^{2k_1} A_2(0)^{2k_2} A_3(0)^{2k_3} \rangle \\ &= \langle A_1(x)^{2k_1} A_1(0)^{2k_1} \rangle \langle A_2(x)^{2k_2} A_2(0)^{2k_2} \rangle \langle A_3(x)^{2k_3} A_3(0)^{2k_3} \rangle \\ &= (2k_1)! (2k_2)! (2k_3)! \Delta^{2k_1} \Delta^{2k_2} \Delta^{2k_3} = (2k_1)! (2k_2)! (2k_3)! \Delta^{2n} . \end{aligned} \quad (23)$$

The last equality was obtained by taking into account the relation $k_1 + k_2 + k_3 = n$.

Next we represent the Kronecker symbol as

$$\delta_{k_1+k_2+k_3, n} = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta} \zeta^{k_1+k_2+k_3-n} , \quad (24)$$

where the contour C includes the origin.

The two point function becomes

$$\langle L(x)L(0) \rangle = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{(n!)^2}{((2n)!)^2} \left(\frac{g^2}{4T} \Delta \right)^{2n} \oint_C \frac{d\zeta}{\zeta^{n+1}} \left(\sum_{k=0}^n \frac{(2k)!}{(k!)^2} \zeta^k \right)^3. \quad (25)$$

We then evaluate the sum

$$\sum_{k=0}^n \frac{(2k)!}{(k!)^2} \zeta^k = \frac{1}{\sqrt{1-4\zeta}} + \zeta^{n+1} F(\zeta), \quad (26)$$

where $F(\zeta)$ is holomorphic about the origin, $\zeta = 0$. Note the factor of ζ^{n+1} in front of $F(\zeta)$. Because of this, only the first term in Eq. (26) contributes,

$$\frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta^{n+1}} \left(\frac{1}{\sqrt{1-4\zeta}} + \zeta^{n+1} F(\zeta) \right)^m = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta^{n+1}} \frac{1}{(1-4\zeta)^{m/2}}. \quad (27)$$

Since the function $F(\zeta)$ is complicated, this helps to simplify the problem. For $m = 3$,

$$\frac{1}{\zeta^{n+1}} \frac{1}{(1-4\zeta)^{3/2}} = \frac{1}{\zeta^{n+1}} \sum_{j=0}^{\infty} 2^{2j} \frac{\Gamma(j + \frac{3}{2})}{j! \Gamma(\frac{3}{2})} \zeta^j. \quad (28)$$

Using the residue theorem,

$$\frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta^{n+1}} \frac{1}{(1-4\zeta)^{3/2}} = 2^{2n} \frac{\Gamma(n + \frac{3}{2})}{n! \Gamma(\frac{3}{2})}. \quad (29)$$

Altogether,

$$\langle L(x)L(0) \rangle = \sum_{n=1}^{\infty} \frac{n!}{((2n)!)^2} \frac{\Gamma(n + \frac{3}{2})}{\Gamma(\frac{3}{2})} \left(\frac{g^2}{T} \Delta \right)^{2n}. \quad (30)$$

Using

$$\Gamma\left(n + \frac{3}{2}\right) = \frac{(2n+2)!}{4^{n+1}(n+1)!} \sqrt{\pi}, \quad (31)$$

it is direct to show that Eq. (30) agrees with the combinatoric result of Eq. (17).

C. Wilson loop

For completeness, in this section we compute a Wilson loop for two colors in the tree approximation. We take a Wilson loop which has the same sides in imaginary time, and just tie the ends at $\tau = 0$ and $\tau = 1/T$ together. Since we drop self energy corrections, we can drop the terms from $\tau = 0$ and $1/T$, and concentrate just on gluons exchanged between the two sides, at \vec{x} and 0 .

In general, there is no simple relation between the exponential of the sum of two elements of the Lie algebra, and the product of the corresponding exponentials. In this case, however, path ordering implies that they do factorize:

$$\mathcal{P} \exp \left(\frac{ig}{2T} (A_a(x)\sigma^a - A_b(0)\sigma^b) \right) = \exp \left(\frac{ig}{2T} A_a(x)\sigma^a \right) \exp \left(-\frac{ig}{2T} A_b(0)\sigma^b \right) \quad (32)$$

There are several ways to establish this. One can check it to the first few orders in g , and then use an inductive proof in powers of $A_a(x)$ and $A_b(0)$. A more elegant proof is the following. Let $A_a(x)$ be infinitesimally small, with $A_b(0)$ arbitrary. Then by path ordering, $A_a(x)$ always lies to the left of any $A_b(0)$. For an infinitesimal $A_a(x)$, to linear order in $A_a(x)$ we have $\approx (1 + ig/(2T)A_a(x)\sigma^a) \exp(ig/(2T)A_b(0)\sigma^b)$. We can then write a differential equation for $A_a(x)$, taking the derivative on the left. The solution of this differential equation is the product of the exponentials in Eq. (32).

Thus the Wilson loop is given by

$$\mathcal{W} = \left\langle \frac{1}{N_c} \text{tr} \left[\exp \left(\frac{ig}{2T} A_a(x)\sigma^a \right) \exp \left(-\frac{ig}{2T} A_b(0)\sigma^b \right) \right] \right\rangle. \quad (33)$$

We can now use the relation of Eq. (4),

$$\begin{aligned} \mathcal{W} = & \left\langle \cos \left(\frac{g}{2T} \sqrt{A_a(x)^2} \right) \cos \left(\frac{g}{2T} \sqrt{A_b(0)^2} \right) \right\rangle \\ & + \left\langle \hat{A}_a(x) \hat{A}_a(0) \sin \left(\frac{g}{2T} \sqrt{A_b(x)^2} \right) \sin \left(\frac{g}{2T} \sqrt{A_c(0)^2} \right) \right\rangle. \end{aligned} \quad (34)$$

For general N_c , the Wilson loop factorizes as above, but then we cannot work out the expansion of the element of the Lie algebra.

The first term in Eq. (34) is equal to the correlator of two Polyakov loops. The second term is new, but can be computed by similar means. In a power series expansion, this second term equals

$$\sum_{n=0}^{\infty} \frac{1}{((2n+1)!)^2} \left(\frac{g}{2T} \right)^{4n+2} \langle A_a(x) A_a(0) (A_b(x)^2)^n (A_c(0)^2)^n \rangle. \quad (35)$$

As before, we neglect self energy corrections, so the powers of $A_b(x)$ and $A_c(0)$ must match.

We need to solve a combinatoric problem very similar to that solved previously, namely the number of ways to tie the fields

$$\langle A_a(x) A_a(0) (A_b(x)^2)^n (A_c(0)^2)^n \rangle. \quad (36)$$

together. We denote this as $\mathcal{D}(2n+1, 3)$. It can be computed using our previous results for $\mathcal{D}(2n, 3)$. When $n = 0$, we have just a sum over one index a , so $\mathcal{D}(1, 3) = 3$.

Now consider general n . There are two cases to consider. For the first case, $A_a(x)$ can tie to $A_a(0)$,

$$\langle A_a(x)A_a(0) \rangle \langle (A_b(x)^2)^n (A_c(0)^2)^n \rangle . \quad (37)$$

This leaves the number of ways to tie $\langle (A_b(x)^2)^n (A_c(0)^2)^n \rangle$ together. Because of the sum over the index “ a ”, there are $3 \mathcal{D}(2n, 3)$ ways of tying the fields together.

The second case is if $A_a(x)$ ties to one of the other $A_c(0)$ ’s. There are $2n$ such $A_c(0)$ ’s. Similarly, $A_a(0)$ can then tie to one of the $A_b(x)$ ’s in $2n$ ways, leaving

$$\langle A_a(x)A_c(0) \rangle \langle A_a(0)A_b(x) \rangle \langle A_b(x)A_c(0)(A_d(x)^2)^{n-1}(A_e(0)^2)^{n-1} \rangle \quad (38)$$

The last correlation function is related to $\mathcal{D}(2n-1, 3)$, so in total there are $(2n)^2 \mathcal{D}(2n-1, 3)$ ways to tying the fields together.

Thus we obtain

$$\mathcal{D}(2n+1, 3) = 3 \mathcal{D}(2n, 3) + (2n)^2 \mathcal{D}(2n-1, 3) = 3(2n+1)! + (2n)^2 \mathcal{D}(2n-1, 3) \quad (39)$$

The solution to this equation is

$$\mathcal{D}(2n+1, 3) = (2n+3)(2n+1)! . \quad (40)$$

This satisfies $\mathcal{D}(1, 3) = 3$, and one can check directly that $\mathcal{D}(3, 3) = 30$.

The result is then

$$\mathcal{W} = (1+z) \cosh(z) + (2+z) \sinh(z) \quad , \quad z = \frac{g^2}{16\pi} \frac{1}{xT} . \quad (41)$$

At large distances, $z \rightarrow 0$, and

$$\mathcal{W} \approx 1 + 3z + \dots \approx 1 + \frac{3g^2}{16\pi} \frac{1}{xT} + \dots \quad , \quad x \rightarrow \infty . \quad (42)$$

This is the expected Coulomb term from single gluon exchange.

At short distances, the exponential agrees with the result from the line-line correlator, Eq. (20).

$$\mathcal{W} \approx z \exp(z) \approx \exp \left(+ \frac{g^2}{16\pi} \frac{1}{xT} + \log \left(\frac{g^2}{16\pi xT} \right) \right) + \dots \quad , \quad x \rightarrow 0 . \quad (43)$$

The Wilson loop is similar to the exponential of the Coulomb term, but the coefficient of $\sim 1/(xT)$ differs by a factor of three. It is for this reason that we refer to the behavior at short distances as Coulomb-like. Notice also that in the exponent, there is a factor of $\log(z) \sim \log(g^2/(xT))$, which has no analogy in perturbation theory. We discuss this further in Sec. (V).

IV. LARGE N_c

For two colors the only difficult part of the problem was in taking into account all of the combinatoric factors from contracting powers of $\sum_a (A_a)^2$ together. For more than two colors, there are two difficulties.

The first is the same problem, keeping track of combinatoric powers of $\sum_a (A_a)^2$, where the sum over the color index “ a ” runs from 1 to $N_c^2 - 1$. In this section we show that in the planar limit, taking $N_c \rightarrow \infty$ while holding $g^2 N_c$ fixed, that the combinatorics is *much* simpler even than two colors.

This simplification is special to the limit of large N_c . The other problem is that for finite $N_c > 2$, the symmetric structure constant, d^{abc} , also arises in the expansion of the exponential. For three or more colors, there is no general expression which relates an element of the Lie algebra, in this case A_a^0 , to one in the Lie group. For reasons which are clear after the fact, this complication can be ignored in the planar limit.

In Appendix (A) and (B), we present the results of computations which interpolate between $N_c = 2$ and ∞ , for some of the lowest order coefficients which arise. They demonstrate the correctness of our results for $N_c = 2$ and ∞ , but show that the case of general N_c is not elementary.

While we can compute in the planar limit, we do not find Coulomb-like behavior at short distances. Nevertheless, we present the results of our computation, since they are relatively easy to obtain.

For three or more colors, the loop and anti-loop are not equal. We begin with the case of the correlator between a loop and an anti-loop, as that is more transparent. Order by order in perturbation theory,

$$\langle L^*(x)L(x) \rangle = \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \left(\frac{g}{T} \right)^{2n} \langle \text{tr} [A^n(x)] \text{tr} [A^n(0)] \rangle \quad (44)$$

As for two colors we neglect self energy corrections, so the number of $A(x)$'s and $A(0)$'s must match. The first term, from $n = 1$, vanishes because of the color trace.

For arbitrary N_c , the problem is computing the color traces in the above expression. Let us then consider the possible contractions of powers of $A(x)$ with those of $A(0)$. We can always choose the first $A(x)$ to be left most in the trace. There are then “ n ” ways to connect this $A(x)$ with any of the $A(0)$'s. We can then make this color matrix the left most in its trace as well,

$$\mathcal{D}_n = \langle \text{tr} [A^n(x)] \text{tr} [A^n(0)] \rangle = n \left(\frac{\Delta}{T} \right) \langle \text{tr} [t^a A^{n-1}(x)] \text{tr} [t^a A^{n-1}(0)] \rangle . \quad (45)$$

To understand our results, consider first the case of an Abelian theory. This is much simpler, because then there are no color traces to bother with. For the first pair of photons, the factor of “ n ” is the same as above. For the second pair of photons, there are $n - 1$ ways to tie the second photon $A(x)$ to one of the other $n - 1$ $A(0)$'s. This continues, so that in all there are $n!$ different ways of connecting the photons together. This cancels one of the factors of $1/n!$ in Eq. (44), giving Eq. (2). In the present case, this is

$$\exp \left(+ \frac{e^2}{4\pi} \frac{1}{xT} \right) \quad (46)$$

This result, from single photon exchange, is valid at all distances. At large distances, this equals $\approx 1 + e^2/(4\pi xT) + \dots$, which is precisely the Coulomb term. That is, in the Abelian theory the Coulomb term directly exponentiates.

In the planar limit, however, the combinatoric factors are trivial to compute. We first give an elementary argument, and then a detailed analysis to verify that.

As a check, in Sec. (A) we have computed the results at finite N_c for the first twelve coefficients of the perturbative expansion of the loop anti-loop correlation functions. The results for the first six terms agree with previous results by Burnier, Laine, and Vepsalainen [3] and by Brambilla, Ghiglieri, Petreczky, and Vairo [4]. Note that we only compare the diagrams at tree level, which are much easier than the computations to higher loop order performed by these authors.

Start with Eq. (45), where we have connected one $A(x)$ to one $A(0)$. Consider first attaching the second gluon $A(x)$, which is immediately to the right of t^a in the color trace. Using the double line notation, this gives the planar diagram of Fig. (1).

Suppose, instead, that we had attached the second gluon $A(x)$ to the *third* $A(0)$. This diagram is not planar, and so suppressed by $1/N_c^2$. Indeed, it is clear diagrammatically that

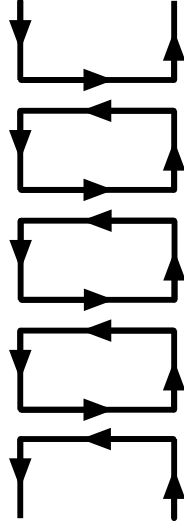


FIG. 1. The planar diagram that dominates at infinite N_c for the loop anti-loop correlation function, in the double line notation. Notice that since we use static gauge, that the ordering of the lines is directly in color space.

the planar diagrams are strictly ordered in color: the second $A(x)$ connects to the second $A(0)$, the third $A(x)$ to the third $A(0)$, and so on.

Thus in the planar limit, the combinatoric factor is simply n . This is in contrast to the Abelian theory, where the corresponding factor is $n!$, and the case of two colors, where the factor is $(n+1)!$.

We now proceed to an equivalent argument, which is useful in computing corrections in $1/N_c^2$ to the planar limit. We start with the identity

$$\text{tr} [t^a M_1] [t^a M_2] = \frac{1}{2} \text{tr} [M_1 M_2] - \frac{1}{2N_c} \text{tr} M_1 \text{tr} M_2 . \quad (47)$$

to reduce

$$D_n = n \frac{\Delta}{2T} \left\langle \text{tr} [A^{n-1}(x) A^{n-1}(0)] - \frac{1}{N_c} \text{tr} [A^{n-1}(x)] \text{tr} [A^{n-1}(0)] \right\rangle . \quad (48)$$

It is not evident, but one can show that the connected part of the second term is suppressed in the limit of large N_c .

Thus we concentrate on the first term. Note that terms of the highest order in N_c originate from contractions as follows. Considering the term with three gluons, this is, graphically,

$$\begin{array}{c} \text{---} \overline{\text{---}} \text{---} \overline{\text{---}} \text{---} \overline{\text{---}} \text{---} \\ | \quad | \quad | \quad | \quad | \\ A(x) A(x) A(x) A(0) A(0) A(0) \end{array} \quad (49)$$

Note that $t^a t^a = C_f \mathbb{I} \approx N_c/2 \mathbb{I}$ at large N_c . Using this, we work gluon by gluon to reduce the trace,

$$\langle \text{tr} [A^{a_1}(x) t^{a_1} A^{a_2}(x) t^{a_2} A^{a_3}(x) t^{a_3} A^{b_1}(0) t^{b_1} A^{b_2}(0) t^{b_2} A^{b_3}(0) t^{b_3}] \rangle \approx \quad (50)$$

$$\Delta(x) \langle \text{tr} [t^{a_1} A^{a_2}(x) t^{a_2} A^{a_3}(x) t^{a_3} A^{b_1}(0) t^{b_1} A^{b_2}(0) t^{b_2} t^{a_1}] \rangle = \quad (51)$$

$$\Delta^2(x) C_f \langle \text{tr} [t^{a_2} A^{a_3}(x) t^{a_3} A^{b_1}(r) t^{b_1} t^{a_2}] \rangle = \quad (52)$$

$$\Delta^2(x) C_f^2 \langle \text{tr} [A^{a_3}(x) t^{a_3} A^{b_1}(x) t^{b_1}] \rangle \approx \Delta^3(x) C_f^3 \text{tr} [\mathbb{I}] = N_c \Delta^3(x) C_f^3. \quad (53)$$

The final factor of N_c comes from the trace of a unit matrix after all contractions are done.

In general,

$$D_n(x) \equiv \langle \text{tr} [A^n(x)] \text{tr} [A^n(0)] \rangle_{N_c \rightarrow \infty} = \frac{n \Delta(x)}{2} N_c \left(\frac{N_c \Delta(x)}{2} \right)^{n-1} = n \left(\frac{N_c \Delta(x)}{2} \right)^n. \quad (54)$$

The sum of the power series gives a Bessel Function I_1 ,

$$\begin{aligned} \langle L^*(x) L(0) \rangle &= 1 + \frac{1}{N_c^2} \sum_{n=2}^{\infty} \frac{1}{n!(n-1)!} \left(\frac{g^2 N_c}{2T} \Delta(x) \right)^n \\ &= 1 + \frac{1}{N_c^2} (z^{1/2} I_1(2z^{1/2}) - z) \quad , \quad z = \frac{g^2 N_c}{8\pi} \frac{1}{xT}. \end{aligned} \quad (55)$$

This is our final result for the loop anti-loop correlator.

Similar arguments can be used to compute the correlator between two Polyakov loops. This is the correlator between a (infinitely heavy) test quark and another test quark. At first the color structure in the planar limit looks different, but one can recognize the following. Take the planar diagram for the loop anti-loop correlator, and flip the line for the anti-loop on its head. This is then the planar diagram for the loop loop correlator.

The only change is then the sign of the charges. For the loop anti-loop correlation function, each term in Eqs. (44) and (55) are positive, as the loop has charge $+ig$, and the anti-loop, charge $-ig$. For the loop loop correlation function, each has charge $+ig$, so there is a factor of $(-1)^n$ in the sum,

$$\begin{aligned} \langle L(x) L(0) \rangle &= 1 + \frac{1}{N_c^2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n!(n-1)!} \left(\frac{g^2 N_c}{2T} \Delta(x) \right)^n \\ &= 1 + \frac{1}{N_c^2} (-z^{1/2} J_1(2z^{1/2}) + z) \quad , \quad z = \frac{g^2 N_c}{8\pi} \frac{1}{xT}. \end{aligned} \quad (56)$$

The propagator is a function only of the absolute value of the spatial separation, x . Then for the correlation functions, only the relative charges matter. Thus the correlation

function between a loop and an anti-loop, with charges ig and $-ig$, is the same as between an anti-loop and a loop, with charges $-ig$ and ig ,

$$\langle L^*(x)L(0) \rangle = \langle L(x)L^*(0) \rangle . \quad (57)$$

Similarly, the correlation function between two loops, each with charge ig , equals that between two anti-loops, each with charge $-ig$,

$$\langle L(x)L(0) \rangle = \langle L^*(x)L^*(0) \rangle . \quad (58)$$

We can now decompose the loop into real and imaginary parts. The above identities imply that there is no correlation between the real and imaginary parts, only between themselves. That for the real part is

$$\langle \text{Re}L(x) \text{Re}L(0) \rangle = 1 + \frac{z^{1/2}}{2N_c^2} [I_1(2z^{1/2}) - J_1(2z^{1/2})] \quad (59)$$

and those for the imaginary parts,

$$\langle \text{Im}L(x) \text{Im}L(0) \rangle = \frac{1}{N_c^2} \left(-z + \frac{z^{1/2}}{2} [I_1(2z^{1/2}) + J_1(2z^{1/2})] \right) \quad (60)$$

The Wilson loop can also be computed directly in the planar limit. By path ordering, we put all powers of $A(x)$ to the left, and all those of $A(0)$ to the right. Neglecting self energy corrections, we expand to “ n ” powers of $igA(x)$, and the same for $-igA(0)$. There is a combinatorial factor of $1/n!$ for each term, $\sim A(x)^n$ and $\sim A(0)^n$. A planar diagram is given by connecting the left most $A(x)$ to the left most $A(0)$. Doing so, we then have to connect the second most left $A(x)$ to the second most left $A(0)$. Continuing in this way, there is no extra factor of “ n ”, and the Wilson loop is given by

$$\mathcal{W} = \frac{1}{N_c} \sum_{n=0}^{\infty} \left(\frac{g}{T} \right)^{2n} \frac{1}{(n!)^2} \langle \text{tr} [A^n(x)A^n(0)] \rangle = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{g^2 N_c}{2T} \Delta(x) \right)^n = I_0(2z^{1/2}) \quad (61)$$

At large distances, $z \rightarrow 0$, the correlation function for the real part of the loop is

$$\langle \text{Re}L(x)\text{Re}L(0) \rangle \approx 1 + \frac{1}{2N_c^2} \left(z^2 + \frac{z^4}{72} \dots \right) = 1 + \frac{1}{8N_c^2} \left(\frac{g^2 N_c}{4\pi} \right)^2 \left(\frac{1}{xT} \right)^2 + \dots \quad (62)$$

The first term, $= 1$, is from the disconnected graph, which dominates at large N_c . The second term is due to connected graphs, which are down by $1/N_c^2$. The coefficient of the term to leading order, $\sim g^2 N_c$, agrees with the large N_c limit of Eq. (19).

For the correlators of the imaginary part of the loop, there are no contributions from disconnected diagrams, only from connected diagrams, which are then necessarily $\sim 1/N_c^2$. For large distances, $z \rightarrow 0$, it begins at $\sim z^3$, which is g^6 :

$$\langle \text{Im}L(x)\text{Im}L(0) \rangle \approx \frac{1}{N_c^2} \left(\frac{z^3}{12} + \frac{z^5}{2880} \dots \right) = \frac{1}{96N_c^2} \left(\frac{g^2 N_c}{4\pi} \right)^3 \left(\frac{1}{xT} \right)^3 + \dots \quad (63)$$

For the Wilson loop, at large distances it behaves as

$$\mathcal{W} \approx 1 + z + \frac{z^2}{4} + \frac{z^3}{36} + \dots \approx 1 + \frac{g^2 N_c}{8\pi} \frac{1}{xT} + \dots \quad (64)$$

The expectation value of the Wilson loop is in and of itself a disconnected diagram, and so all contributions survive in the planar limit at infinite N_c . The leading term, $\sim g^2$, is from single gluon exchange.

What is less obvious is the limit of short distances. For small z ,

$$I_\alpha(z) \rightarrow \frac{e^z}{\sqrt{2\pi z}} \quad , \quad J_1(z) \rightarrow \frac{\sin(z) - \cos(z)}{\sqrt{\pi z}} \quad , \quad (65)$$

we find that the behavior of *all* correlation functions at small distances is not the exponential of $\sim 1/r$, but the exponential of $\sim 1/\sqrt{r}$. For example, for the Wilson loop

$$\mathcal{W} \approx \exp(2\sqrt{z}) \approx \exp\left(2\sqrt{\frac{g^2 N_c}{8\pi}} \sqrt{\frac{1}{xT}}\right) \quad (66)$$

The behavior at short distances can also be derived directly from the sum over n . At short distances, z is large, and one can show that the dominant behavior is given by asymptotically large n . To show this, one can use an effective action. For the Wilson line at large N_c , it is

$$\frac{1}{(n!)^2} z^n = \exp(\mathcal{S}_{eff}(n)) \quad , \quad (67)$$

where

$$\mathcal{S}_{eff}(n) = -2 \log(n!) + n \log z \approx_{n \rightarrow \infty} -2(n \log(n) - n) + n \log z \quad (68)$$

At large n , the stationary point of the effective action $\partial \mathcal{S}_{eff}/\partial n = 0$, is given by $n_0 = \sqrt{z}$. Then $\mathcal{S}_{eff}(n_0) \sim 2n_0 = 2\sqrt{z}$, in accord with Eq. (66). The same method can be used to derive the short distance behavior of the other correlation functions. This is perfectly reasonable: the behavior at large distances, when z is small, is determined by the first few orders in perturbation theory. That at short distances, when z is large, is determined by the asymptotically large orders of perturbation theory.

V. CONCLUSIONS

In this paper we computed the correlators of Polyakov (and Wilson) loops at tree level, as functions of $g^2 N_c/(xT)$. In this section we suggest that the non-Coulombic behavior at short distances in Eq. (66) is an artifact of the planar limit at tree level.

We first review, on a heuristic level, why one expects Coulombic behavior at short distances. As we work in imaginary time, the length in the time direction is $\beta = 1/T$. Now consider the limit of $\beta \rightarrow \infty$. The discussion is similar for either the Wilson loop or the correlation function between the loop and anti-loop (or loop loop). For simplicity we discuss the Wilson loop, since all contributions persist in the planar limit.

Single gluon exchange gives a contribution $\sim g^2 N_c \beta/x$. Two gluon exchange gives rise to two terms. The two gluons can be moved independently up and down the time axis, which gives the square of the first term, $\sim (g^2 N_c \beta/x)^2$. There are also terms where the two gluons do not move independently, $\sim (g^2 N_c)^2 \beta/x$. The latter represents corrections at one loop order to the potential at tree level.

For our purposes we can ignore perturbative corrections to the tree level potential. The leading term is from the exchange of “ n ” gluons, which move independently up and down the time axis. These one gluon kernels are identical, and so there is an associated combinatoric factor of $1/n!$. This then generates the exponential of the potential at tree level. Because we assume that $\beta \rightarrow \infty$ at the outset, by dimensional analysis the coefficient of any term $\sim \beta$ is necessarily Coulombic, $\sim 1/x$. This heuristic analysis underlies extensive studies in perturbation theory, beginning with Ref. [7].

A more modern approach uses effective theories, such as potential non-relativistic QCD [4]. For the Wilson loop, or the loop anti-loop correlator, at short distances one relates these correlation functions to two fields, for a color singlet and a color octet. The effective theory approach directly allows the exponentiation of the potential in each channel, as the two point function is equal to the exponential of the potential, times the expectation value of the associated Polyakov loop in either the singlet or octet representations.

The same analysis can be carried out for the loop loop correlator. The operators which enter are those for operators with two indices in the fundamental representation. There are two irreducible representations, either symmetric or anti-symmetric in the two indices. (For example, for three colors these are the sextet and anti-triplet representations, respectively.)

For each representation, the two point function of the loop loop correlator is then related to the exponential of the associated potential, times the expectation value of the Polyakov loop in the associated representation.

This analysis using effective theory applies in weak coupling for arbitrary N_c , in the limit that $\beta \rightarrow \infty$. This is different from our computations, where the parameter $g^2 N_c \beta / x$ is kept finite throughout. Consequently, we believe that our result is peculiar to the planar approximation. We do not understand why the result at short distances is weaker than Coulombic in the planar limit.

We conclude by discussing the implications of our results for the work of Lo, Friman, Redlich, and Sasaki [5]. Usually what is computed in non-Abelian gauge theories are the expectation values of (renormalized) Polyakov loops in different representations. The process of renormalization is not simple on the lattice [8–11]. While the self energy corrections to the loop vanish with dimensional regularization, they do not on the lattice. Instead, they give rise to terms $\sim \exp(-C_{\mathcal{R}} g^2 / (aT))$, where “ a ” is the lattice spacing, and $C_{\mathcal{R}}$ the Casimir for the loop in an irreducible representation \mathcal{R} . By the character expansion, products of loops at the same point reduce to a linear sum of loops in irreducible representations [9].

In Refs. [5] Lo, Friman, Kaczmarek, Redlich, and Sasaki study the susceptibility of Polyakov loops for three colors. What is especially interesting is the difference between the susceptibilities for the real and imaginary parts of the loop.

For local fields, there is no concern about the measurement of such susceptibilities. If one measures $\phi(x)\phi(0)$, then by the Operator Product Expansion there are new ultraviolet divergences as $x \rightarrow 0$. However, these are local terms. A susceptibility involves the average of each field over the volume of space (or space-time), \mathcal{V} . Thus while new divergences will arise, accompanied by powers of, *e.g.*, the lattice spacing “ a ”, in the end they will be overwhelmed by an overall factor of $1/\mathcal{V}$. That is, the divergences associated with bringing two operators together does not affect the measurement of the susceptibility.

Polyakov loops, however, are non-local operators, and on the lattice, have exponential divergences. Thus ultraviolet divergences, which are proportional to $\sim \exp(+\#/(aT)^m)$, will overwhelm factors of the volume, $1/\mathcal{V}$, in the continuum limit, as $aT \rightarrow 0$. For operators which are dominated by Coulomb behavior at small distances, $m = 1$. In the planar limit we find $m = 1/2$, but we believe this is special to large N_c . For the imaginary part of the loop, the leading perturbative correction is from three gluon exchange, Eq. (63), and so

then $m = 3$. It appears necessary to eliminate the effects of these divergences before one can extract susceptibilities of the Polyakov loops which are free from effects of the lattice cut-off.

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Appendix A: Perturbative expansion at arbitrary N_c

In the perturbative expansion of the correlation functions, we need

$$\mathcal{D}_n = \langle \text{tr} [A^n(x)] \text{tr} [A^n(0)] \rangle = c_n \left(\frac{\Delta(x)}{T} \right)^n. \quad (\text{A1})$$

By brute force, the first eleven coefficients are

$$c_2 = \frac{N_c^2}{2} - \frac{1}{2}, \quad (\text{A2})$$

$$c_3 = \frac{3N_c^3}{8} - \frac{15N_c}{8} + \frac{3}{2N_c}, \quad (\text{A3})$$

$$c_4 = \frac{N_c^4}{4} - \frac{7N_c^2}{4} + 6 - \frac{9}{2N_c^2}, \quad (\text{A4})$$

$$c_5 = \frac{5N_c^5}{32} - \frac{25N_c^3}{32} + \frac{35N_c}{8} - \frac{75}{4N_c} + \frac{15}{N_c^3}, \quad (\text{A5})$$

$$c_6 = \frac{3N_c^6}{32} + \frac{15N_c^4}{32} - \frac{99N_c^2}{16} + \frac{45}{8} + \frac{225}{4N_c^2} - \frac{225}{4N_c^4}, \quad (\text{A6})$$

$$c_7 = \frac{7N_c^7}{128} + \frac{49N_c^5}{32} - \frac{2597N_c^3}{128} + \frac{6657N_c}{64} - \frac{735}{4N_c} - \frac{2205}{16N_c^3} + \frac{945}{4N_c^5}, \quad (\text{A7})$$

$$c_8 = \frac{N_c^8}{32} + \frac{35N_c^6}{16} - \frac{875N_c^4}{32} + \frac{1545N_c^2}{8} - 903 + \frac{3675}{2N_c^2} - \frac{2205}{2N_c^6}, \quad (\text{A8})$$

$$c_9 = \frac{9N_c^9}{512} + \frac{621N_c^7}{256} - \frac{9639N_c^5}{512} + \frac{9873N_c^3}{128} - \frac{11745N_c}{16} + \frac{90153}{16N_c} - \frac{59535}{4N_c^3} + \frac{8505}{2N_c^5} + \frac{5670}{N_c^7}, \quad (\text{A9})$$

$$c_{10} = \frac{5N_c^{10}}{512} + \frac{75N_c^8}{32} + \frac{3705N_c^6}{512} - \frac{94975N_c^4}{256} + \frac{95265N_c^2}{32} - \frac{31725}{4} - \frac{155925}{8N_c^2} + \frac{439425}{4N_c^4} - \frac{212625}{4N_c^6} - \frac{127575}{4N_c^8}, \quad (\text{A10})$$

$$c_{11} = \frac{11N_c^{11}}{2048} + \frac{4235N_c^9}{2048} + \frac{94743N_c^7}{2048} - \frac{2051555N_c^5}{2048} + \frac{9832823N_c^3}{1024} - \frac{15454395N_c}{256} + \frac{13476375}{64N_c} - \frac{4459455}{32N_c^3} - \frac{12006225}{16N_c^5} + \frac{8575875}{16N_c^7} + \frac{779625}{4N_c^9}, \quad (\text{A11})$$

$$c_{12} = \frac{3N_c^{12}}{1024} + \frac{1749N_c^{10}}{1024} + \frac{92169N_c^8}{1024} - \frac{1365573N_c^6}{1024} + \frac{2542947N_c^4}{256} - \frac{9823527N_c^2}{128} + \frac{39051045}{64} - \frac{45721665}{16N_c^2} + \frac{33960465}{8N_c^4} + \frac{36018675}{8N_c^6} - \frac{5145525}{N_c^8} - \frac{5145525}{4N_c^{10}}. \quad (\text{A12})$$

For two colors, all c_n vanish for odd n , while for even n ,

$$c_n = (n+1)! \frac{1}{2^{2n-2}}, \quad (\text{A13})$$

which agrees with our previous analysis: the factor of $(n+1)!$ is $\mathcal{N}(n, 3)$ in Eq. (16).

Appendix B: Corrections at large N_c

From the above results, we could not deduce the general result for arbitrary n and N_c . However, at large N_c the first two terms satisfy

$$c_n \approx n \left(\frac{N_c}{2} \right)^n \left(1 + \frac{1}{24 N_c^2} n(n-1)(n(n-1) - 26) + O\left(\frac{1}{N_c^4} \right) \right) \quad (\text{B1})$$

The first term, $= n(N_c/2)^n$, agrees with our combinatorial analysis above. The second term is admittedly a guess, satisfied by $c_2 \dots c_{12}$.

We now perform the exercise to assume that this guess is correct, and use it to compute the leading correction in $1/N_c^2$.

The leading order in $1/N_c^2$ to the correlator for the loop anti-loop correlation function is

$$\frac{1}{N_c^4} \left(\frac{7}{8} z^2 + \frac{1}{12} z^2 I_0(2\sqrt{z}) + \frac{z^{3/2}}{24} (z - 26) I_1(2\sqrt{z}) \right). \quad (\text{B2})$$

For the loop loop correlator, one changes $z \rightarrow -z$,

$$\frac{1}{N_c^4} \left(\frac{7}{8} z^2 + \frac{1}{12} z^2 J_0(2\sqrt{z}) - \frac{z^{3/2}}{24} (z + 26) J_1(2\sqrt{z}) \right). \quad (\text{B3})$$

Guessing the corrections to higher order is not trivial. Those $\sim 1/N_c^4$ are proportional to

$$\frac{9}{16} n^6 - \frac{4567}{240} n^5 + \frac{4289}{16} n^4 - \frac{96733}{48} n^3 + \frac{67251}{8} n^2 - \frac{179387}{10} n + 14553. \quad (\text{B4})$$

when $n > 4$.

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- [1] E. Gava and R. Jengo, [Phys.Lett. **B105**, 285 \(1981\)](#).
 - [2] P. B. Arnold and L. G. Yaffe, [Phys.Rev. **D52**, 7208 \(1995\)](#), [arXiv:hep-ph/9508280 \[hep-ph\]](#).
 - [3] Y. Burnier, M. Laine, and M. Vepsalainen, [JHEP **1001**, 054 \(2010\)](#), [arXiv:0911.3480 \[hep-ph\]](#).
 - [4] N. Brambilla, J. Ghiglieri, P. Petreczky, and A. Vairo, [Phys.Rev. **D82**, 074019 \(2010\)](#), [arXiv:1007.5172 \[hep-ph\]](#).
 - [5] P. M. Lo, B. Friman, O. Kaczmarek, K. Redlich, and C. Sasaki, [Phys.Rev. **D88**, 014506 \(2013\)](#), [arXiv:1306.5094 \[hep-lat\]](#); [Phys.Rev. **D88**, 074502 \(2013\)](#), [arXiv:1307.5958 \[hep-lat\]](#).
 - [6] S. Borsanyi, Z. Fodor, S. D. Katz, A. Psztor, K. K. Szab, et al., (2015), [arXiv:1501.02173 \[hep-lat\]](#).

- [7] W. Fischler, [Nucl.Phys. **B129**, 157 \(1977\)](#); T. Appelquist, M. Dine, and I. Muzinich, [Phys.Lett. **B69**, 231 \(1977\)](#); [Phys.Rev. **D17**, 2074 \(1978\)](#).
- [8] O. Kaczmarek, F. Karsch, P. Petreczky, and F. Zantow, [Phys.Lett. **B543**, 41 \(2002\)](#), [arXiv:hep-lat/0207002 \[hep-lat\]](#).
- [9] A. Dumitru, Y. Hatta, J. Lenaghan, K. Orginos, and R. D. Pisarski, [Phys.Rev. **D70**, 034511 \(2004\)](#), [arXiv:hep-th/0311223 \[hep-th\]](#).
- [10] S. Gupta, K. Huebner, and O. Kaczmarek, [Phys.Rev. **D77**, 034503 \(2008\)](#), [arXiv:0711.2251 \[hep-lat\]](#).
- [11] A. Mykkanen, M. Panero, and K. Rummukainen, [JHEP **1205**, 069 \(2012\)](#), [arXiv:1202.2762 \[hep-lat\]](#).